RELATIVISTIC EFFECTS IN THE GLOBAL POSITIONING SYSTEM

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Abstract

The Global Positioning System relies on a system of synchronized atomic clocks orbiting the earth that emit signals which upon reception at any point on earth, are used to compute the position coordinates of the point. These are subject to frequency shifts because of their altitude and gravitation and require that relativistic effects be taken into account for their accurate functioning. This article looks at the relativistic basis of the primary corrections that are to be made when using GPS, and also shows some numerical calculations to give an idea of the errors that may accumulate if the relevant corrections are not made.

Introduction

The GPS has a segmented configuration, consisting of Space, Control and User segments respectively. It is in the Space segment that 24 satellites carrying atomic clocks are orbiting the earth with a period of 12 hours. The configuration is such that there are four satellites in each of six orbital planes inclined at 55° with respect to earth's equatorial plane, distributed so that from any point on the earth, four or more satellites are almost always above the local horizon [1]. The Control segment consists of multiple monitoring stations on earth that track the satellites, receive and transmit navigational updates, synchronize the clocks to within nanoseconds of each other and predict ephemerides of each satellite [2]. The User segment is essentially the users who are able to receive signals from the orbiting clocks and compute their positions, local time etc.

The role of relativity is evident in the very process that is used to determine the user position. We consider four synchronized clocks at positions $\mathbf{r_j}$ that emit pulses at times $\mathbf{t_j}$, where j = {1, 2,3,4}. The transmission of these pulses can be thought of events in space-time that correspond to a phase reversal of the circularly polarized electromagnetic signals [1]. These signals are then received on earth by a user who needs to know her position and time coordinates given by **r** and **t** respectively. At the user end, the time lapse between the emission and detection of the signal is used to compute how far each satellite is from the user. This is done by solving the following equation,

$$c^2 (t - t_j)^2 = |\mathrm{r} - \mathrm{r}_j|^2, \; j = 1, 2, 3, 4 \qquad (1)$$

where c is the speed of light. This is followed by the method of trilateration to determine the user's position **r**. Essentially after the distances of three satellites from the user are known, three spheres are constructed centered on each satellite respectively. The intersecting point of these three spheres gives the position of the user. The fourth atomic clock is used to make up for the inaccuracies of the user's clock. We can see the importance of eliminating timing errors, by calculating the effect on distance calculation as a result of an error in the reported time lapse.

$$c\Delta \mathrm{t} = \Delta \mathrm{r} \qquad (2)$$

From (2) it can be seen that a Δt of 1ns corresponds to a Δr of 0.3m. This shows that even small differences in the frequencies of clocks on earth and in satellites can add up to significant errors. We will quantify this further in the next sections.

Gravitational frequency shift

Gravitational blue shift experienced by the signals emitted by the satellite clocks is a major effect that needs to be accounted for in GPS. The main reason for this effect is the reduced curvature of space-time experienced by the satellite clocks due to their increased distance from earth. The calculations below show how this corresponds to the satellite clocks running faster than identical clocks on earth, and consequently accumulating a time error that needs to be adjusted.

We will use the Schwarzschild metric approximation to analyze the clocks in GPS[3]. The Schwarzschild metric and the corresponding differential proper time are given by,

$$ds^2 = \left(1 - \frac{2M}{r}\right)dt^2 - \frac{dr^2}{\left(1-\frac{2M}{r}\right)} - r^2 d\varphi^2 \quad (3)$$

$$d\tau^2 = \left(1 - \frac{2M}{r}\right)dt^2 - \frac{dr^2}{\left(1-\frac{2M}{r}\right)} - r^2 d\varphi^2 \quad (4)$$

The value of *c* is taken as 1 in the above expressions. Assuming that the clocks are constant distance from earth, we can take dr to be 0. We divide equation (4) by $(dt)^2$ where dt is the incremental coordinate time of a clock at infinity. Now, we can write two different equations, one for $d\tau_{sat}$ - the satellite proper time and $d\tau_e$ – the proper time for the clock on earth.

$$\left(\frac{d\tau_{sat}}{dt}\right)^2 = \left(1 - \frac{2M}{r_{sat}}\right) - {r_{sat}}^2 \left(\frac{d\varphi}{dt}\right)^2 \quad (5)$$

$$\left(\frac{d\tau_e}{dt}\right)^2 = \left(1 - \frac{2M}{r_e}\right) - {r_e}^2 \left(\frac{d\varphi}{dt}\right)^2 \quad (6)$$

Writing **rdφ/dt** as **v,** the tangential velocity of the clock along the circular path, and dividing the above two equations, we get:

$$\left(\frac{d\tau_{sat}}{d\tau_e}\right)^2 = \frac{1-\frac{2M}{r_{sat}} - v_{sat}^2}{1-\frac{2M}{r_e} - v_e^2} \quad (7)$$

Equation (7), gives the ratio of the proper times of the clocks at two different locations from the centre of the earth. We can calculate the effect due to gravitational blue-shift alone by taking v to be 0 for both the clocks and apply the following approximation when taking the square root of equation (7):

$(1+d)^n = 1+ nd$, if $|d|<<1$ and $|nd|<<1$ (8)

We obtain the following expression for the stationary clocks,

$$\frac{d\tau_{sat}}{d\tau_e} \approx 1 - \frac{M}{r_{sat}} + \frac{M}{r_e} - \frac{M^2}{r_{sat}\, r_e} \quad (9)$$

The cross term can be neglected for its relatively small magnitude and we are left with,

$$\frac{d\tau_{sat}}{d\tau_e} \approx 1 - \frac{M}{r_{sat}} + \frac{M}{r_e} \quad (10)$$

The above expression is an estimate of the fractional difference in rates between the stationary clocks at two different positions above the earth. The above ratio is of the order of 10^{-10}. To be precise, for every one second on earth the satellite clock is ahead by 5.2 X 10^{-10}s. This means that in one day i.e. 86400s, this asynchrony will amount to 45 microseconds. We can use equation (2) to see that this would lead to an accumulated error in position by 13.5km in one day. Hence, the atomic clocks in satellites need to be adjusted in advance to make up for their faster rates. However, this is not the only correction to be made. We now move on to other effects and see how they should be accounted for.

Time dilation

Time dilation i.e. slowing down of clocks under motion is one of the foremost consequences of special relativity. In GPS too, the atomic clocks on the satellites are moving with respect to an observer on earth. This means that we should take into account the velocities in equation (7) for calculating the fractional time difference. The velocities of the satellite in orbit as well as the clock on earth – v_{sat} and v_e respectively – are computed using Newtonian mechanics. Knowing that the period of the satellite is 12 hours and assuming a circular orbit, the velocity of the satellite is given by,

$$v_{sat} = \frac{2\pi r_{sat}}{T} \qquad (11)$$

Similarly, the velocity of a clock on the equator can be calculated by noticing that the it completes one cycle of rotation around the earth's axis in one day or 86,400s.

$$v_e = \frac{2\pi r_e}{86{,}400} \qquad (12)$$

Substituting equations (11) and (12) in (7) and repeating the analysis as in the case of stationary clocks, we obtain the following expression,

$$\frac{d\tau_{sat}}{d\tau_e} \approx 1 - \frac{M}{r_{sat}} + \frac{M}{r_e} - \frac{v_{sat}^2}{2} + \frac{v_e^2}{2} \qquad (13)$$

Again, substitution of values gives an offset of 39 microseconds for one day, i.e. the satellite clock is slower by 39 microseconds in one day. We notice that time dilation and gravitational blue shift are two opposing effects, the former making the satellite clocks tick faster whereas the later slows them down. Before a clock is launched into orbit, the net effect is taken into account and the frequency of the clocks is slowed down [4].

The Sagnac Effect

Equation (1) implies an inertial reference frame in which the user determines her position coordinates. However, the rotation of earth about its axis has an additional effect that needs to be considered when attempting to synchronize clocks in a rotating frame of reference. We analyze this by considering a transformation from an inertial frame with Minkowskian space-time to a rotating frame of reference. Neglecting gravitational potentials, the metric in cylindrical coordinates in an inertial frame is written as,

$$-ds^2 = -(c\,dt)^2 + dr^2 + r^2 d\varphi^2 + dz^2 \qquad (14)$$

We now transform this to another frame rotating at ω_e, with coordinates {t',r',φ',z'} given by

$$t = t',\ r = r',\ \varphi = \varphi' + \omega_E t',\ z = z'. \qquad (15)$$

The transformed metric is written as,

$$-ds^2 = -\left(1 - \frac{\omega_E^2 r'^2}{c^2}\right)(cdt')^2 + 2\omega_E r'^2 d\varphi' dt' + (d\sigma')^2 \qquad (16)$$

where, $(d\sigma')^2 = (dr')^2 + (r'd\varphi')^2 + (dz')^2$ (17)

Light travels along a null worldline, so we set ds^2 equal to zero, and solve for *cdt'* keeping terms of only up to first order,

$$cdt' = d\sigma' + \frac{\omega_E r'^2 d\varphi'}{c} \qquad (18)$$

The quantity $r'2d\varphi'/2$ is just the infinitesimal area dA'_z in the rotating coordinate system swept out by a vector from the rotation axis to the light pulse, and projected onto a plane parallel to the equatorial plane[1]. Thus, the total time required for light to traverse some path is

$$\int dt' = \int \frac{d\sigma'}{c} + \frac{2\omega_E}{c^2} \int dA'_z \qquad (19)$$

A fixed observer on earth, not taking into account the rotation would only consider the first term in the synchronization of her clock network. On the other hand, the contribution due to the second term is significant as shown below and therefore must be accounted for when performing clock synchronizations.

For earth, $2\omega_E/c^2 = 1.6227 \times 10^{-21}$ s m^{-2} and the equatorial radius is a = 6,378,137 m, so the area is πa^2= 1.27802×1014 m^2. Thus, the last term in equation (19) gives,

$$\frac{2\omega_E}{c^2} \int dA'_z = 207.4\, ns \qquad (20)$$

From an underlying inertial reference frame, this can be considered as the additional time required by light to reach the moving reference point [1].

Conclusion

It has been shown that without proper adjustments in clock frequencies to make up for the relativistic effects, GPS will experience serious inaccuracies. Relativity manifests itself in the form of constancy of speed of light, gravitational frequency shifts, time dilation and rotational effects. The purpose of this project was to provide an overview of the major considerations that are taken into account when designing GPS. There are other secondary effects such as signal propagation delay, spatial curvature on the geoid, gravitation from other solar system bodies and perturbation in satellite orbits that could also be taken into account in a more advanced analysis to obtain more refined results.

References

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