To Love or Not to Love

Disclaimer: Romeo and Juliet bear no resemblance to any character, living or dead, depicted in any of Shakespeare's or Taylor Swift's works.

Math, unlike people, doesn't lie - so if you fall in love, model your romance with these simple, do-it-yourself set of differential equations:

- There is a boy: Romeo [R].
- There is a girl: Juliet [J].

We are mathematicians. We have no emotions. Let's analyze their ♥.

We start with defining some terms.

$\frac{dR}{dt}$ = This is Romeo's response i.e. the rate of change of Romeo's love for Juliet. If this is a positive number, Romeo keeps falling in love. If this value is negative, his love decreases/hatred increases.

$\frac{dJ}{dt}$ = The same as before.

Now let's discuss a very simple statement:

$$\frac{dR}{dt} = \boldsymbol{a}R + \boldsymbol{b}J$$

If we hold time still at time = t or May 4^{th} 2010 2:30 pm. When Romeo's love for Juliet is R, Juliet's love for Romeo is J and the change in Romeo's love after, for instance, a minute i.e. at 2:31 pm is dependent on variables R and J. Of course, time here is continuous so we will consider an infinitesimally small change in time instead of a 1 minute time interval. But that's just a detail.

<u>*Case 1 – Where is Juliet?*</u>

What happens when $\frac{dR}{dt}$ is not dependent on Juliet's love at all. This is an example of selfless, unconditional love or just a high school crush. Romeo loves Juliet without her knowing, or caring, or maybe even existing, for instance if Romeo has fallen in love with a story book character. This is the case where **b** = 0. Romeo's love would keep increasing ($\frac{dR}{dt} > 0$). If **a** is a positive constant and **a** > 1, love would actually keep increasing at an exponential rate till it probably drives poor Romeo crazy so R = 0. If **a** lies between 0 and 1, then love would still increase, but at a decreasing rate. If a is negative $\mathbf{a} < 0$, then this is not love, its hate. A highly negative number like -20 may result in hateful Romeo murdering his evil, loud neighbor Juliet.

Case 2 – Juliet is clingy.

Generally, Romeos dislike obsessive Juliets. In this case **b** would be a large negative number. Any advances by Juliet would drive Romeo away, that is $\frac{dR}{dt}$ would become smaller and then finally negative where he starts cringing at the sight of Juliet.

However, if Romeo is needy, he might enjoy a clingy Juliet thus **b** is positive and they both keep falling in **love** as J increases, then R increases and then woah - so much love!

Well, we understand the simple equation now. The same theory applies for Juliet and we get a set of differencial equations:

$$\frac{dR}{dt} = \boldsymbol{a}R + \boldsymbol{b}J$$

$$\frac{dJ}{dt} = \boldsymbol{c}R + \boldsymbol{d}J$$

This can be rewritten as matrices:

$$\begin{pmatrix} R' \\ J' \end{pmatrix} = \begin{pmatrix} \boldsymbol{a} & \boldsymbol{b} \\ \boldsymbol{c} & \boldsymbol{d} \end{pmatrix} \begin{pmatrix} R \\ J \end{pmatrix}$$

Where $\begin{pmatrix} \boldsymbol{a} & \boldsymbol{b} \\ \boldsymbol{c} & \boldsymbol{d} \end{pmatrix} = M$

Most people, when they get into a relationship, wonder how long it will last. Some want it to last forever. We will now teach them how to evaluate stability. A love life is stable if (R, J) that is Romeo and Juliet's love for each other, reaches a value and stays there.

Friendly advice from a Math major with a non-existent love life: here we ask you to disregard all the movie dialogues which imply that love increases every day. Love might increase everyday but it will probably give rise to an unstable fixed point. An unstable fixed point is like a hill-top. A stone may reach the highest point but then it also rolls down. A stable fixed point, on the other hand, is like a valley. A stone will stay at the bottom of this valley regardless of the initial conditions. Thus, unstable fixed points give rise to relationships that do not work.

Assume a solution to the set of equations:

$$R = Ae^{\lambda t} \text{ and } J = Be^{\lambda t}$$

And so:

$$R' = A\lambda e^{\lambda t} \text{ and } J' = B\lambda e^{\lambda t}$$

$$\begin{pmatrix} R' \\ J' \end{pmatrix} = \begin{pmatrix} \boldsymbol{a} & \boldsymbol{b} \\ \boldsymbol{c} & \boldsymbol{d} \end{pmatrix} \begin{pmatrix} R \\ J \end{pmatrix}$$

$$\begin{pmatrix} A\lambda e^{\lambda t} \\ B\lambda e^{\lambda t} \end{pmatrix} = \begin{pmatrix} \boldsymbol{a} & \boldsymbol{b} \\ \boldsymbol{c} & \boldsymbol{d} \end{pmatrix} \begin{pmatrix} Ae^{\lambda t} \\ Be^{\lambda t} \end{pmatrix}$$

$$A\lambda e^{\lambda t} = (\boldsymbol{a}A + \boldsymbol{b}B)e^{\lambda t} \text{ and } B\lambda e^{\lambda t} = (\boldsymbol{c}A + \boldsymbol{d}B)e^{\lambda t}$$

$$0 = (\boldsymbol{a} - \lambda)A + \boldsymbol{b}B \text{ and } 0 = \boldsymbol{c}A + (\boldsymbol{d} - \lambda)B$$

Therefore,

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \boldsymbol{a} - \lambda & \boldsymbol{b} \\ \boldsymbol{c} & \boldsymbol{d} - \lambda \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}$$

This can only happen if:

$$\begin{vmatrix} \boldsymbol{a} - \lambda & \boldsymbol{b} \\ \boldsymbol{c} & \boldsymbol{d} - \lambda \end{vmatrix} = 0$$

And so,

$$(\boldsymbol{a} - \lambda)(\boldsymbol{d} - \lambda) - \boldsymbol{bc} = 0$$

$$\boldsymbol{ad} - (\boldsymbol{a} + \boldsymbol{d})\lambda + \lambda^2 - \boldsymbol{bc} = 0$$

$$\lambda^2 - (\boldsymbol{a} + \boldsymbol{d})\lambda + (\boldsymbol{ad} - \boldsymbol{bc}) = 0$$

Where

$$(\boldsymbol{a} + \boldsymbol{d}) = Trace\ of\ matrix\ M\ (TrM)$$

and

$$(\boldsymbol{ad} - \boldsymbol{bc}) = determinant\ of\ mtrix\ M\ (DetM)$$

Therefore,

$$\lambda = \left(\frac{-(TrM) \pm \sqrt{(TrM)^2 - 4(DetM)}}{2}\right)$$

This implies that we get two values of λ: λ_1 *and* λ_2

Case 1: λ_1, λ_2 *are real and distinct*

$$R = X_1 e^{\lambda_1 t} + X_2 e^{\lambda_2 t}$$

$$J = Y_1 e^{\lambda_1 t} + Y_2 e^{\lambda_2 t}$$

For stability we will need $\lambda_1, \lambda_2 < 0$

Case 2: *Complex conjugate roots*

Eigenvalues:

$$\lambda_{1,2} = \alpha + \iota\beta$$

Eigenvectors:

$$R = A_1 e^{(\alpha+\iota\beta)t} + A_2 e^{(\alpha+\iota\beta)t}$$

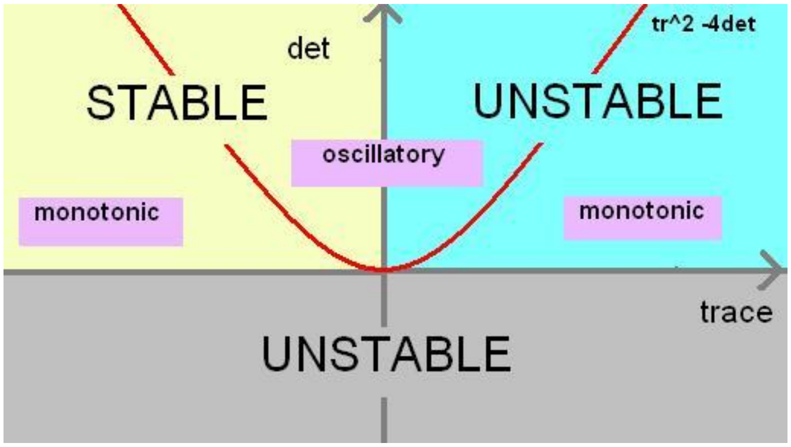
and

$$J = B_1 e^{(\alpha+\iota\beta)t} + B_2 e^{(\alpha+\iota\beta)t}$$

For stability we need:

$$\alpha \equiv Re(\lambda) < 0$$

A useful summary of stable and unstable points:



Source: Dr. Adnan Khan's *Mathematical Biology* Resource Material

Further Modifications to the Model:

The model elucidated above can further by modified to make it apply for love that is timeless. This is done by adding an exponential term to both equations - exp($-\gamma t$) is added so both Romeo and Juliet's love dies as they age.

Another interesting case would be the modification of the model for the case of a love triangle – this is achieved simply by adding a third variable S.

Wafa Veljee